Num. IV.

Ejusdem

De ſummis ſerierum reciprocarum ex poteſtatibus numerorum naturalium ortarum Diſſertatio altera:

in qua

eædem ſummationes ex fonte maxime diverſo derivantur.

§. I.

Poſtquam ante complures annos ſummas ſerierum in hac forma generali contentarum $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} +$ &c. in infinitum, ſi n fuerit numerus par poſitivus, ſimulque etiam harum ſerierum, ſi n fuerit numerus impar $1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} +$ &c. in infinitum, ope quadraturæ circuli exhibuiſſem, atque oſtendiſſem, ſummam perpetuo per eandem peripheriæ circuli dignitatem, quam exponens n indicet, exprimi: argumentum hoc acutiſſimis Geometris tantopere placuit, ut id non ſolum maxime probarent, verum etiam ipſi

ipſi ſtudium atque operam ad easdem ſummas methodis ſibi familiaribus eruendas impenderent. Ego quoque ab illo tempore multum fui occupatus in alia via inveſtiganda, quæ eodem deduceret, non tam ut veritatem inventam magis confirmarem, quam ut limites analyſcos in hujus generis ſeriebus tractandis ulterius extenderem.

§. II.

Methodus, quæ me ad iſtarum ſerierum ſummationem manuduxit, utique erat nova, & in ejusmodi inſtituto plane non uſitata: nitebatur enim in reſolutione æquationis infinitæ, cujus omnes radices, quarum numerus erat infinitus, noſſe oportebat. Contemplatus ſum namque hanc æquationem in infinitum excurrentem:

$$x = s - \frac{s^3}{6} + \frac{s^5}{120} - \frac{s^7}{5040} + \frac{s^9}{362880} - \text{\&c.}$$

quæ continet relationem inter arcum circuli s & ejus ſinum x, ſinu toto poſito $= 1$. Cum autem eidem ſinui x innumerabiles arcus tam affirmativi quam negativi reſpondeant; hoc modo iſtius æquationis radices innumeras a poſteriori eram conſecutus; & quoniam coefficientes cujusque æquationis a radicibus pendent, ex comparatione iſtorum coefficientium cum radicibus æquationis perveni ad ſummas ſerierum ante memoratarum.

§. III.

Statim quidem facile perſpexi, iſtam methodum tutam eſſe, atque ad veritatem perducere non poſſe, niſi conſtaret

æquationem illam gradus infiniti alias radices, præter eas, quas natura ſinuum mihi ſuppeditaverat, in ſe non complecti: Quamvis enim intelligerem, alias radices reales, præter aſſignatas, in illa æquatione non contineri, tamen merito dubitare conveniebat, an non radices imaginariæ ineſſent, quod ſi eveniſſet, cunctæ ſummationes, quas inde elicui, cum veritate conſiſtere non potuiſſent. In hoc porro dubio confirmabar, cum ſimili modo ex arcu elliptico ejus ſinum ſeu applicatam reſpondentem per ſeriem expresſiſſem: quanquam enim pariter innumerabiles arcus elliptici præſto erant, qui ad eundem ſinum referantur, tamen ex iis nullas ſerierum ſummas, quæ veritati eſſent conſentaneæ, deducere licuerat: cujus incommodi ratio procul dubio in pluribus ac fortaſſe infinitis radicibus imaginariis erat poſita, quæ in illam æquationem ex ellipſi formatam ingrediuntur.

§. IV.

Quoniam igitur illo tempore nullam habueram demonſtrationem, qua mihi certo conſtaret, æquationem inter arcum circuli s ejusque ſinum x radicibus imaginariis omnino carere: ſummas ſerierum inventas ad trutinam veritatis examinare cœpi: ac primo quidem ſtatim deprehendi, hanc methodum eandem ſummam ſeriei:

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \text{ \&c.}$$

præbere, quam jam pridem Leibnitius aſſignaverat quæ convenientia ſatis declarabat, ſi quas æquatio radices imaginarias contineret, earum ſummam neceſſario eſſe $= 0$. Deinde

ſeries

ſeries altiorum poteſtatum ita examinavi, ut ſummas hac methodo inventas cum ſummis, quas paulo ante per approximationem erueram, compararem; in quo negotio denuo conſenſum deprehendi. Atque ob has rationes penitus confirmabar, nullis omnino radicibus imaginariis æquationem illam, quæ me ad omnes illas ſummas perduxerat, eſſe inquinatam; ſicque non dubitavi iſtas ſummas tanquam veriſſimas producere.

§. V.

In hac opinione autem prorſus me corroborabat alia methodus mere analytica, cujus ope deinceps per ſolas integrationes eandem ſummam hujus ſeriei $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ &c. elicueram, ſimilique fere modo hanc eandem ſummam demonſtratam dedit Vir longe acutiſſimus Nicolaus Bernoulli, in ſchediasmate Tomo IX. Comment. Accad. Petropol. inſerto. Quanquam autem hoc modo calculus analyticus ad easdem ſummas omnes eruendas traduci poſſe videbatur; tamen neque ego neque quisquam alius pari calculo ad poteſtates altiores pertingere potuimus. Quod me fere eo impulit, ut crederem, aliam viam præter reſolutionem æquationis infinitæ non patere, quæ omnium poteſtatum ſummas junctim præberet.

§. VI.

Curam hanc fere penitus oblitam nuper in me renovaverunt litteræ a Viro Celeb. Daniele Bernoulli acceptæ, in quibus mihi easdem dubitandi rationes circa meam methodum objicit, ſimulque ſignificavit Celeberrimum Crame-

rum

rum iisdem dubiis præpediri, quominus methodum meam probare queat. Iſtæ igitur admonitiones amiciſſimæ eo me redegerunt, ut univerſum negotium de novo ruminarer, atque cum in methodi bonitate demonſtranda, tum in alia via easdem ſeries ſummandi exquirenda laborarem. In utroque igitur inſtituto voti plene compos factus, duplex negotium in hac diſſertatione abſolvam. Primum ſcilicet demonſtrabo, in æquatione infinita ſupra commemorata radices imaginarias nullas contineri, hincque de ſummarum inde deductarum veritate amplius dubitari non licere. Secundo vero loco novam ac non ſolum a prima maxime diverſam methodum, ſed etiam ad plurima alia luſtranda campum aperientem proponam, quæ per ſolas integrandi regulas totum negotium conficiat.

§. VII.

Demonſtrationem primum promiſſam impetravi ex reſolutione hujus binomii $a^n + b^n$ in factores ſuos reales. Unus quisque enim hujus binomii factor continetur in hac forma $aa - 2ab \operatorname{cof} A \frac{(2k-1)\pi}{n} + bb$ atque omnes factores obtinentur, ſi loco $2k-1$ ſucceſſive omnes numeri impares minores exponente n ſubſtituantur: &, ſi n fuerit numerus impar, tum præter hos factores trinomiales adjungi debet factor ſimplex $a+b$. Quod ſi habeatur reſiduum $a^n - b^n$ ejus primo factor eſt ſimplex $a-b$; factores reliqui trinomiales reales continentur in hac forma $aa - 2ab \operatorname{cof} A \frac{2k\pi}{n}$

$+ bb$;

$+ bb$; cunctique hujus generis factores obtinentur, ſi loco $2k$ ſucceſſive omnes numeri pares (cyphra excepta) minores exponente n ſcribantur: &, ſi n ipſe fuerit numerus par, præterea factor adjici debet ſimplex $a + b$. Hoc igitur modo omnes omnino formulæ $a^n \pm b^n$ factores reales exhiberi poterunt, quorum omnium productum hanc ipſam formulam reddat. Ceterum hic notandum eſt π denotare ſemi circumſerentiam circuli, cujus radius $= 1$, ſeu π eſſe angulum duobus rectis æqualem.

§. VIII.

Hinc jam a priori aſſignare poſſum omnes radices ſeu factores hujus expreſſionis infinitæ

$$s - \frac{s^3}{1.2.3.} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.\ldots.7} + \frac{s^9}{1.2.3\ldots9} - \&c.$$

Hæc enim expreſſio æquivalet iſti $\dfrac{e^{s\sqrt{-1}} - e^{-s\sqrt{-1}}}{2\sqrt{-1}}$

denotante e numerum, cujus logarithmus eſt $= 1$. & cum ſit $e^z = \left(1 + \frac{z}{n}\right)^n$ exiſtente n numero infinito, reducetur

expreſſio infinita propoſita ad hanc: $\dfrac{\left(1 + \frac{s\sqrt{-1}}{n}\right)^n - \left(1 - \frac{s\sqrt{-1}}{n}\right)^n}{2\sqrt{-1}}$:

cujus primum factor ſimplex eſt s, quem quidem ipſa ſeriei inſpectio monſtrat. Ad reliquos factores eruendos comparo hanc expreſſionem cum forma $a^n - b^n$ erit $a = 1 + \frac{s\sqrt{-1}}{n}$; & $b = 1 - \frac{s\sqrt{-1}}{n}$; hincque $aa + bb = 2 -$

$\frac{2ss}{nn}$ & $2ab = 2 + \frac{2ss}{nn}$. Quilibet ergo factor continebitur in hac forma $2 - \frac{2ss}{nn} - 2\left(1 + \frac{ss}{nn}\right) \operatorname{coſ} A \frac{2k\pi}{n}$ hincque omnes omnino factores emergent, ſi pro $2k$ ſucceſſive omnes numeri pares in infinitum ſubſtituantur, propterea quod n denotat numerum infinitum.

§. IX.

Cum autem n ſit numerus infinitus, erit arcus $\frac{2k\pi}{n}$ infinite parvus, quoad $2k$ fiat quoque infinitus numerus, minor tamen quam n. Erit ergo $\operatorname{coſ} A \frac{2k\pi}{n} = 1 - \frac{2kk\pi\pi}{nn}$, unde factor generalis tranſit in hanc formam: $-\frac{4ss}{nn} + \frac{4kk\pi\pi}{nn}$ ex quo, termino cognito ad unitatem reducto, naſcitur factor $1 - \frac{ss}{kk\pi\pi}$, qui loco k ſubſtitutis ſucceſſive omnibus numeris 1, 2, 3, &c. in infinitum præbet omnes factores. Quodſi autem k in infinitum abeat, ut $2k$ ad n acquirat rationem finitam, tum ob $\operatorname{coſ} A \frac{2k\pi}{n} < 1$ termini $\frac{ss}{nn}$ præ unitate non deſtructa evaneſcent, fietque factor $1 - \frac{2k\pi}{n}$ conſtans, ideoque in computum non venit, quod in eo arcus s non continetur.

§. X.

§. X.

Hoc ergo pacto nacti ſumus omnes factores formulæ propoſitæ

$$s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3\ldots 7} + \&c.$$

quæ adeo exacte æqualis erit producto ex his infinitis factoribus conſtante:

$$s\left(1 - \frac{ss}{\pi\pi}\right)\left(1 - \frac{ss}{4\pi\pi}\right)\left(1 - \frac{ss}{9\pi\pi}\right)\left(1 - \frac{ss}{16\pi\pi}\right) \&c.$$

ex quorum cum coefficientibus terminorum ſeriei comparatione ſtatim ſequuntur ſummæ ſerierum

$$1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \frac{1}{6^m} + \&c.$$

ſi m denotet numerum quemcunque parem, neque igitur amplius de earum veritate dubitari poteſt.

§. XI.

Simili modo ſi conſideremus hanc ſeriem:

$$1 - \frac{ss}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.3\ldots 6} + \frac{s^8}{1.2.3\ldots 8} - \&c.$$

ea reducetur ad hanc formam $\dfrac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n + \left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2}$

denotante n numerum infinitum. Diviſores ergo binomii $\left(1 + \frac{s\sqrt{-1}}{n}\right)^n + \left(1 - \frac{s\sqrt{-1}}{n}\right)^n$ ſimul erunt diviſores ſeriei propoſitæ, & quidem omnes. Comparata hac forma cum $a^n + b^n$ erit $a = 1 + \frac{s\sqrt{-1}}{n}$; $b = 1 - \frac{s\sqrt{-1}}{n}$; $aa + bb = 2 - \frac{2ss}{nn}$

$\frac{2ss}{nn}$ & $2ab = 2 + \frac{2ss}{nn}$, unusquisque ergo formulæ propoſitæ diviſor continetur in hac expreſſione $2\left(1 - \frac{ss}{nn}\right) - 2\left(1 + \frac{ss}{nn}\right)\operatorname{coſ} A \frac{(2k-1)\pi}{n}$, vel hac $2\left(1 - \operatorname{coſ} A \frac{(2k-1)\pi}{n}\right) - \frac{2ss}{nn}\left(1 + \operatorname{coſ} A \frac{(2k-1)\pi}{n}\right)$. Cum autem in diviſore tantum ad incognitam reſpiciatur, erit diviſor quilibet $1 - \frac{ss\left(1 + \operatorname{coſ} A \frac{(2k-1)\pi}{n}\right)}{nn\left(1 - \operatorname{coſ} A \frac{(2k-1)\pi}{n}\right)}$ facto termino cognito unitati æquali, quia in ſerie ipſa primus terminus eſt $= 1$.

§. XII.

Ob n autem numerum infinitum erit $1 + \operatorname{coſ} A \frac{(2k-1)\pi}{n} = 2$ & $1 - \operatorname{coſ} A \frac{(2k-1)\pi}{n} = \frac{(2k-1)^2 \pi\pi}{2nn}$ ex quo unusquisque diviſor erit $1 - \frac{4ss}{(2k-1)^2 \pi\pi}$; & ſi loco $2k-1$ omnes numeri impares in infinitum usque ſubſtituantur, orientur omnes diviſores ſeriei propoſitæ $1 - \frac{ss}{1.2.} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.\ldots.6} +$ &c. quæ ergo erit productum ex his factoribus numero infinitis:

$$\left(1 - \frac{4ss}{\pi\pi}\right)\left(1 - \frac{4ss}{9\pi\pi}\right)\left(1 - \frac{4ss}{25\pi\pi}\right)\left(1 - \frac{4ss}{49\pi\pi}\right) \text{ \&c.}$$

ex quorum comparatione cum ipſa ſerie, omnes ſeries poteſtatum ſummantur ut ante. Atque ſic demonſtratum eſt

æqua-

æquationes illas infinitas, quas illo tempore tractavi, alias radices non habere, præter eas, quas ex natura ſinuum & coſinuum a poſteriore ſum aſſecutus.

§. XIII.

Demonſtrata methodi, cujus ope antehac iſtiusmodi ſerierum ſummas asſignavi, bonitate, progredior ad alteram methodum ab hac prorſus diverſam explicandam, quæ ex ſolis calculi integralis principiis petita ſine ullis ambagibus earundem ſerierum ſummas mira facilitate ſuppeditat. Nititur autem iſta methodus duobus theorematis, quorum demonſtrationem in diſſertatione ſuperiori *de inventione integralium ſi quantitati variabili poſt integrationem definitus valor tribuatur*, dedi; unde ea ſine demonſtratione depromam. Primum ita ſe habet:

„ Formulæ differentialis $\frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx$ integrale ita
„ ſumtum, ut evaneſcat poſito $x=0$, ſi poſt integratio-
„ nem ponatur $x=1$, dabit hunc valorem $\frac{\pi}{q\,\text{ſin}\,A\frac{p\pi}{q}}$,
„ denotante π ſemicircumferentiam circuli, cujus radius $=1$,
„ in quo circulo ſimul ſinum arcus $\frac{p\pi}{q}$ capi pono.

Alterum theorema huic fere ſimile ita ſe habet:

„ Formulæ differentialis hujus $\frac{x^{p-1}-x^{q-p-1}}{1-x^q}dx$ integrale

„ ita

„ ita ſumtum, ut evaneſcat poſito $x = 0$, ſi in eo poſt „ integrationem ponatur $x = 1$ dabit hunc valorem

$$\frac{\pi \operatorname{coſ} A \frac{p\pi}{q}}{q \operatorname{ſin} A \frac{p\pi}{q}}, \text{ ſeu } \frac{\pi}{q \operatorname{tang} A \frac{p\pi}{q}}.$$

Demonſtrationes horum theorematum ordine plano procedunt, primum enim ſecundum regulas conſuetas iſtarum formularum integralia generatim inveſtigavi, iisque inventis loco variabilis x unitatem poſui. Quo facto ad ſeriem ſinuum finitam adeo pertigi, quæ quia arcus in progreſſione arithmetica progrediebantur, ſummationem admittebat, hasque ipſas expresſiones præbebat.

§. XIV.

Sumamus jam priorem formulam integralem

$\int \frac{x^{p-1} + x^{q-p-1}}{1 + x^q} dx$, quæ in ſeriem reſoluta dabit duplicem progresſionem geometricam:

$$+ \int dx \, (x^{p-1} - x^{q+p-1} + x^{2q+p-1} - x^{3q+p-1} + \&c.)$$
$$+ \int dx \, (x^{q-p-1} - x^{2q-p-1} + x^{3q-p-1} - x^{4q-p-1} + \&c.)$$

Hujus igitur integrale ita acceptum ut evaneſcat poſito $x = 0$, ita per ſeriem exprimetur:

$$\frac{x^p}{p} + \frac{x^{q-p}}{q-p} - \frac{x^{q+p}}{q+p} - \frac{x^{2q-p}}{2q-p} + \frac{x^{2q+p}}{2q+p} + \frac{x^{3q-p}}{3q-p}$$

$-$ &c. Quodſi nunc ponamus $x = 1$, erit per theorema prius hujus ſeriei:

$\frac{1}{p}$

$$\frac{1}{p}+\frac{1}{q-p}-\frac{1}{q+p}-\frac{1}{2q-p}+\frac{1}{2q+p}+\frac{1}{3q-p}-\frac{1}{3q+p}-\frac{1}{4q-p}$$

$+$ &c. ſumma $= \dfrac{\pi}{q \operatorname{ſin} A \frac{p\pi}{q}}$.

§. XV.

Simili modo altera formula integralis:

$\int \dfrac{x^{p-1} - x^{q-p-1}}{1-x^q} dx$ per ſeriem integrata dabit:

$$\frac{x^p}{p}-\frac{x^{q-p}}{q-p}+\frac{x^{q+p}}{q+p}-\frac{x^{2q-p}}{2q-p}+\frac{x^{2q+p}}{2q+p}-\frac{x^{3q-p}}{3q-p}+$$

&c. Quamobrem per alterum theorema ſi ponamus $x = 1$, erit hujus ſeriei

$$\frac{1}{p}-\frac{1}{q-p}+\frac{1}{q+p}-\frac{1}{2q-p}+\frac{1}{2q+p}-\frac{1}{3q-p}+\frac{1}{3q+p}$$

$-$ &c. ſumma $= \dfrac{\pi \operatorname{coſ} A \frac{p\pi}{q}}{q \operatorname{ſin} A \frac{p\pi}{q}}$; dummodo fuerint p & q numeri affirmativi, atque $q > p$, id quod in ſequentibus ſemper aſſumi oportet: alioquin enim integrale hoc modo ſumtum non evaneſceret poſito $x = 0$.

§. XVI.

Sit $\dfrac{p}{q} = s$, atque ſeriebus inventis per q multiplicatis habebimus has binas ſeries ad ſummam finitam reductas:

π

$$\frac{\pi}{\text{ſin}\,A\,s\pi} = \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \frac{1}{3+s} - \&c.$$

$$\frac{\pi\,\text{coſ}\,A\,s\pi}{\text{ſin}\,A\,s\pi} = \frac{1}{s} - \frac{1}{1-s} + \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} - \frac{1}{3-s} + \frac{1}{3+s}$$

— &c. harumque ſerierum ſummæ veræ erunt, quicunque numerus per s indicetur, ſive rationalis ſive irrationalis, ſicque lex continuitatis non amplius infringitur ut ante, ubi loco p & q numeros integros accipi oportebat. Quin etiam hæ ſummæ a veritate non recedunt, etiamſi pro s numeri unitate majores ponantur. Si enim ſit $s = 1$ vel quilibet numerus integer, tum ſeries fiunt infinitæ ob unum terminum in infinitum abeuntem, ſimul vero ſummæ exhibitæ ob ſin $A\,s\pi = 0$ in infinitum excreſcunt. Hinc iſtæ ſummæ tam late patent, ut nullam prorſus reſtrictionem requirant.

§. XVII.

Ex his ſeriebus generalibus jam deducuntur ſeries pro quadratura circuli cum Leibnizii, tum Gregorii aliæque innumerabiles; quarum præcipuas hic exhibere lubet.

Sit $q = 2$ & $p = 1$ erit ſin $A\,\frac{\pi}{2} = 1$ & coſ$A\frac{\pi}{2} = 0$

hincque ſequentes ſeries naſcuntur

$$\frac{\pi}{2} = 1 + 1 - \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{7} + \&c. \text{ ſive}$$

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \&c. \text{ atque}$$

$\frac{o\pi}{2}$

$$\frac{0\,\pi}{2} = 1 - 1 + \tfrac{1}{3} - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{7} + \&c.$$

quarum illa eſt ſeries Leibniziana, hæc vero ſe ſponte prodit.

Sit $q = 3$ & $p = 1$, erit ſin A $\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ & coſ A $\frac{\pi}{3} = \frac{1}{2}$, unde ſequentes ſeries naſcuntur

$$\frac{2\,\pi}{3\sqrt{3}} = 1 + \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{11} + \tfrac{1}{13} + \&c.$$

$$\frac{\pi}{3\sqrt{3}} = 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{10} - \tfrac{1}{11} + \tfrac{1}{13} - \&c.$$

Sit $q = 4$; $p = 1$, erit ſin A $\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ & coſ A $\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ hincque ſequentes oriuntur ſeries

$$\frac{\pi}{2\sqrt{2}} = 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{13} - \&c.$$

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} - \&c.$$

Sit $q = 6$; $p = 1$, erit ſin A $\frac{\pi}{6} = \frac{1}{2}$ coſ A $\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ unde ſequentes proficiſcuntur ſeries

$$\frac{\pi}{3} = 1 + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} - \tfrac{1}{23} + \&c.$$

$$\frac{\pi}{2\sqrt{3}} = 1 - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} - \tfrac{1}{23} + \&c.$$

Haſque ſeries omnes prior quoque methodus ſuppeditaverat.

§. XVIII.

Quoniam igitur vidimus ſummam hujus ſeriei

$$\frac{1}{s}+\frac{1}{1-s}-\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}+\frac{1}{3-s}-\&c.\ \text{eſſe}=\frac{\pi}{\text{ſin}\,A\,\pi s}$$

& hujus

$$\frac{1}{s}-\frac{1}{1-s}+\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}-\frac{1}{3-s}+\&c.\ \text{eſſe}=\frac{\pi\,\text{coſ}\,A\pi s}{\text{ſin}\,A\,\pi s}$$

quicunque valor litteræ s tribuatur; manifeſtum eſt, easdem æqualitates locum habere, ſi loco s ponatur $s+ds$, vel quod eodem redit, ſi illæ ſeries cum ſuis ſummis differentientur, poſita quantitate s variabili. Quare cum ſit $d.\,\text{ſin}\,A\,\pi s= \pi\,ds\,\text{coſ}\,A\pi s$, & $d.\,\text{coſ}\,A\,\pi s=-\pi\,ds\,\text{ſin}\,A\,\pi s$, erit ſumtis differentialibus, atque ubique per $-ds$ diviſione facta:

$$\frac{\pi\pi\,\text{coſ}\,A\,\pi s}{(\text{ſin}\,A\,\pi s)^2}=\frac{1}{ss}-\frac{1}{(1-s)^2}-\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}-\frac{1}{(3-s)^2}-\&c.$$

$$\frac{\pi\pi}{(\text{ſin}\,A\,\pi s)^2}=\frac{1}{ss}+\frac{1}{(1-s)^2}+\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}+$$

&c. Quodſi ergo loco s reſtituatur $\frac{p}{q}$ & utrinque per qq dividatur, prodibunt ſequentes ſeries ſummatæ.

$$\frac{\pi\pi\,\text{coſ}\,A\frac{p\pi}{q}}{qq(\text{ſin}\,A\frac{p\pi}{q})^2}=\frac{1}{pp}-\frac{1}{(q-p)^2}-\frac{1}{(q+p)^2}+\frac{1}{(2q-p)^2}+\frac{1}{(2q+p)^2}+\&c.$$

$$\frac{\pi\pi}{qq(\text{ſin}\,A\frac{p\pi}{q})^2} = \frac{1}{pp} + \frac{1}{(q-p)^2} + \frac{1}{(q+p)^2} + \frac{1}{(2q-p)^2} + \frac{1}{(2q+p)^2} + \&c.$$

§. XIX.

Ponamus eſſe $q = 2$ & $p = 1$, erit ſin $A\frac{\pi}{2} = 1$ & coſ $A\frac{\pi}{2} = 0$, unde ſequentes prodibunt ſeries:

$$0 = 1 - 1 - \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{5^2} - \frac{1}{7^2} + \&c.$$

$$\frac{\pi\pi}{4} = 1 + 1 + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.$$

quarum prioris veritas per ſe patet, poſterior vero huc redit:

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \&c.$$

Sit $q = 3$ & $p = 1$, erit ſin $A\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, & coſ $A\frac{\pi}{3} = \frac{1}{2}$, unde naſcentur iſtæ binæ ſeries.

$$\frac{2\pi\pi}{27} = 1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \frac{1}{10^2} + \&c.$$

$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{10^2} + \&c.$$

Sit $q = 4$ & $p = 1$, erit ſin $A\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ & coſ $A\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ atque hinc orientur hæ duæ ſeries:

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \&c.$$

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \&c.$$

Sit $q = 6$ & $p = 1$, erit ſin A $\frac{\pi}{6} = \frac{1}{2}$, & coſ A $\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ quo caſu obtinebuntur iſtæ ſeries:

$$\frac{\pi\pi}{6\sqrt{3}} = 1 - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{17^2} - \frac{1}{19^2} + \&c.$$

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \&c.$$

Atque ex his ſeriebus non difficulter derivantur binæ illæ principales, quas præcedentis methodi ope in hoc genere elicui; nempe:

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \&c.$$

$$\frac{\pi\pi}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \&c.$$

§. XX.

Quo facilius ſummæ altiorum poteſtatum per continuatam differentiationem inveniri queant; ſummas atque ſeries ſeorſim differentiemus. Sit igitur $\frac{\pi}{\text{ſin A} \pi s} = P$ & $\frac{\pi \text{ coſ A} \pi s}{\text{ſin A} \pi s} = Q$; habebimusque ſequentes ſummationes per differentialia cujusque gradus ipſorum P & Q expreſſas:

$+ P =$

$$+P = \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \&c.$$

$$+Q = \frac{1}{s} - \frac{1}{1-s} + \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} - \frac{1}{3-s} + \&c.$$

$$\frac{-dP}{1ds} = \frac{1}{ss} - \frac{1}{(1-s)^2} - \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} - \frac{1}{(3-s)^2} - \&c.$$

$$\frac{-dQ}{1ds} = \frac{1}{ss} + \frac{1}{(1-s)^2} + \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} + \frac{1}{(3-s)^2} + \&c.$$

$$\frac{+ddP}{1.2\,ds^2} = \frac{1}{s^3} + \frac{1}{(1-s)^3} - \frac{1}{(1+s)^3} - \frac{1}{(2-s)^3} + \frac{1}{(2+s)^3} + \frac{1}{(3-s)^3} - \&c.$$

$$\frac{+ddQ}{1.2\,ds^2} = \frac{1}{s^3} - \frac{1}{(1-s)^3} + \frac{1}{(1+s)^3} - \frac{1}{(2-s)^3} + \frac{1}{(2+s)^3} - \frac{1}{(3-s)^3} + \&c.$$

$$\frac{-d^3P}{1.2.3\,ds^3} = \frac{1}{s^4} - \frac{1}{(1-s)^4} - \frac{1}{(1+s)^4} + \frac{1}{(2-s)^4} + \frac{1}{(2+s)^4} - \frac{1}{(3-s)^4} - \&c.$$

$$\frac{-d^3Q}{1.2.3\,ds} = \frac{1}{s^4} + \frac{1}{(1-s)^4} + \frac{1}{(1+s)^4} + \frac{1}{(2-s)^4} + \frac{1}{(2+s)^4} + \frac{1}{(3-s)^4} + \&c.$$

Generaliter ergo habebitur iſta ſummatio:

$$\frac{\mp d^{n-1}P}{1.2.3\ldots(n-1)ds^{n-1}} = \frac{1}{s^n} \mp \frac{1}{(1-s)^n} - \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \mp \frac{1}{(3-s)^n} + \&c.$$

$$\frac{\mp d^{n-1}Q}{1.2.3\ldots\ldots(n-1)ds^{n-1}} = \frac{1}{s^n} \mp \frac{1}{(1-s)^n} + \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \mp \frac{1}{(3-s)^n} + \&c.$$

ubi ſigna ſuperiora valent, ſi n fuerit numerus impar, inferiora vero, ſi n ſit numerus par.

§. XXI.

Ad ſummas igitur has actu determinandas, neceſſe eſt, ut valores differentialium cujuscunque ordinis quantitatum P & Q eruamus: quod ut facilius & ſuccinctius fieri poſſit, ponamus ſin $A\pi s = x$, & coſ $A\pi s = y$; eritque $P = \frac{\pi}{x}$ & $Q = \frac{\pi y}{x}$. Porro vero erit $dx = \pi y ds$, & $dy = -\pi x ds$, unde per regulas differentiationis ſequentes obtinentur valores

$$P = \frac{\pi}{x}$$

$$-\frac{dP}{ds} = \frac{\pi\pi}{xx} \cdot y$$

$$+\frac{ddP}{ds^2} = \frac{\pi^3}{x^3}\ (y^2+1)$$

$$-\frac{d^3 P}{ds^3} = \frac{\pi^4}{x^4}(y^3 + 5y)$$

$$+\frac{d^4 P}{ds^4} = \frac{\pi^5}{x^5}(y^4 + 18y^2 + 5)$$

$$-\frac{d^5 P}{ds^5} = \frac{\pi^6}{x^6}(y^5 + 58y^3 + 61y)$$

$$+\frac{d^6 P}{ds^6} = \frac{\pi^7}{x^7}(y^6 + 179y^4 + 479y^2 + 61)$$

$$-\frac{d^7 P}{ds^7} = \frac{\pi^8}{x^8}\left(y^7 \begin{matrix} +6.1 \\ +3.179 \end{matrix} y^5 \begin{matrix} +4.179 \\ +5.479 \end{matrix} y^3 \begin{matrix} +2.479 \\ +7.61 \end{matrix} y\right)$$

&c.

ex quarum ultima expreſſione ſimul lex conſtat, cujus ope ex differentiali cujusvis ordinis, differentiale ſequentis ordinis formari poteſt. Hincque iſtius ſeriei

$$\frac{1}{s^n} \pm \frac{1}{(1-s)^n} - \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \pm \frac{1}{(3-s)^n} - \&c.$$

ſumma pro quovis exponentis n valore aſſignabitur.

§. XXII.

Simili modo reperientur valores differentialium cujuscunque ordinis alterius quantitatis Q eritque

$$+ Q = \frac{\pi}{x} \cdot y$$

$$-\frac{dQ}{ds} = \frac{\pi\pi}{xx} \cdot 1$$

$$+\frac{ddQ}{ds^2} = \frac{\pi^3}{x^3} \cdot 2y$$

— *d*

$$-\frac{d^3 Q}{ds^3} = \frac{\pi^4}{x^4}(4yy+2)$$

$$+\frac{d^4 Q}{ds^4} = \frac{\pi^5}{x^5}(8y^3+16y)$$

$$-\frac{d^5 Q}{ds^5} = \frac{\pi^6}{x^6}(16y^4+88y^2+16)$$

$$+\frac{d^6 Q}{ds^6} = \frac{\pi^7}{x^7}(32y^5+416y^3+272y)$$

$$-\frac{d^7 Q}{ds^7} = \frac{\pi^8}{x^8}(64y^6+1824y^4+2880y^2+272)$$

$$+\frac{d^8 Q}{ds^8} = \frac{\pi^9}{x^9}\left(2.64y^7 + \begin{matrix}4.1824\\+7.64\end{matrix}\,y^5 + \begin{matrix}6.2880\\+5.1824\end{matrix}\,y^3 + \begin{matrix}8.272\\+3.2880\end{matrix}\,y\right)$$

&c.

Lex progreſſionis hic pariter patet, cujus ope, quo usque libuerit, progredi licet: atque hinc exhiberi poterit ſumma omnis ſeriei poteſtatum, quæ in hac forma continetur:

$$\frac{1}{s^n} \mp \frac{1}{(1-s)^n} + \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \mp \frac{1}{(3-s)^n} + \&c.$$

In his autem ſeriebus non ſolum continentur omnes poteſtatum ſeries, quas præcedens methodus ſuppeditaverat, ſed inſuper innumerabiles aliæ. Quin etiam ipſa hæc methodus ad plura alia præclara eruenda idonea videtur.

Num. V.